Brief Introduction of k-Domination Number on Cartesian Product of Complete Graphs

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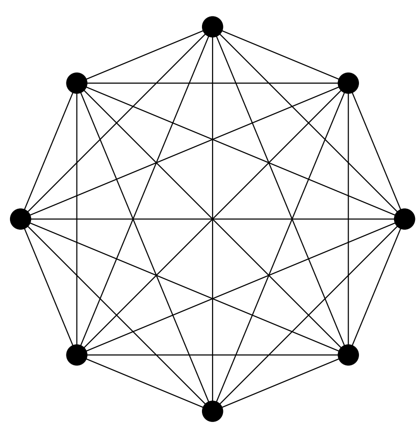
The project which my coworkers and I worked on during the summer at Florida International University is about the k-domination on the Cartesian product of Complete graphs. It is an open question because there is no single algebraic ways or fixed methods to find any k-domination of the Cartesian product of complete graphs.

1 Definition

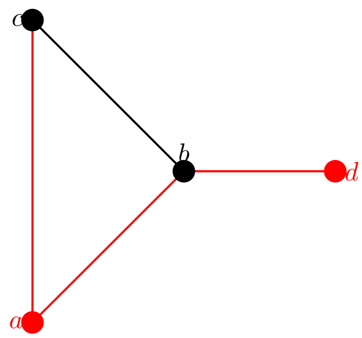
1.1 Graph and Complete Graph

A graph, G, consists an vertex set, $V(G)$ and an edge set $E(G)$. We use $G = \{V(G), E(G)\}$ to denote it. If two vertices are connected by an edge, then we say these two vertices are adjacent to each other. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by an edge. We use K_n to denote a complete graph with n vertices. The graph below is a complete graph with 8 vertices, K_8.

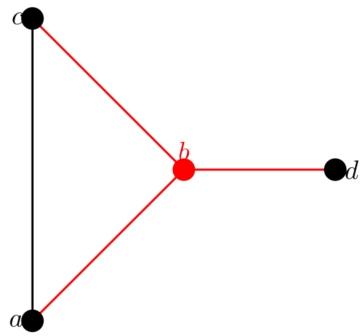


1.2 k-Dominating Set and k-Domination

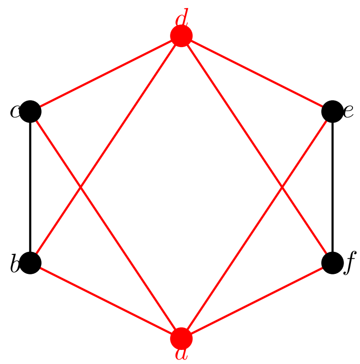
If D is a subset of the vertex set $V(G)$, such that for any vertex, $v \in V(G) - D$ is adjacent to at least k vertices in D, then D is a k-dominating set of G. k-Domination number of a graph G, $\gamma_k(G)$, is the number of vertices in the smallest k-dominating set of G. For the vertex set $D = \{a, d\}$ of the graph below, c and b are not in $\{a, d\}$, but each is adjacent to at least one vertex in D (c is adjacent to a and b is adjacent to a and d). Hence, $\{a, d\}$ is a 1-dominating set of the graph.



However, it is not the smallest 1-dominating set we can find since $\{b\}$ is also a 1-dominating set. Because one vertex is necessary to build a 1-dominating set of this graph, the 1-domination number, $\gamma_1(G)$, is 1.

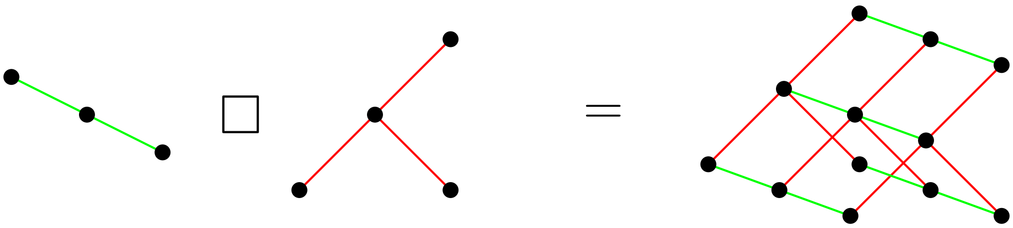


For the graph below, $\{a, d\}$ is a 2-dominating set and it is also a smallest 2-dominating set, so $\gamma_2(G) = 2$.



1.3 Cartesian Product of Graphs

The Cartesian product of two graphs is still a graph. The vertex set of Cartesian product of graphs G and H, $V(G \square H)$, is the Cartesian product of the vertex sets of graphs G and H, $V(G) \times V(H)$. Two vertices, (u, u'), (v, v') in $G \square H$ are adjacent to each other if $u = v$ and u' is adjacent to v' in graph H, or $u' = v'$ and u is adjacent to v in graph G. The image below is an example.



The graph we care about is the Cartesian product of the complete graphs $K_n \square K_m$, and $\gamma_k(K_n \square K_m)$ is the focus of our research.

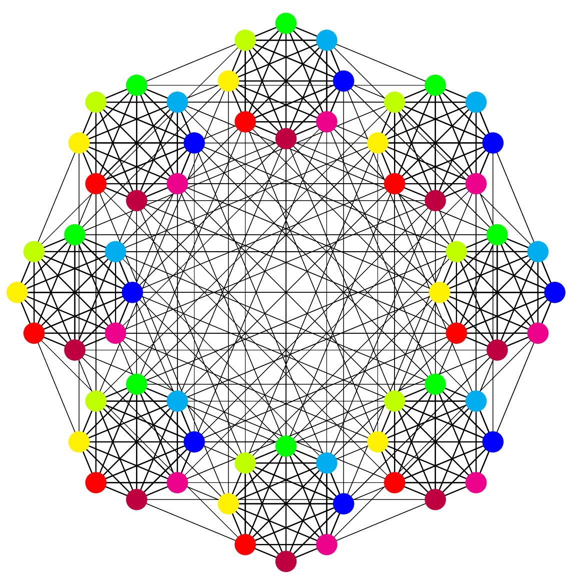
2 Motivation

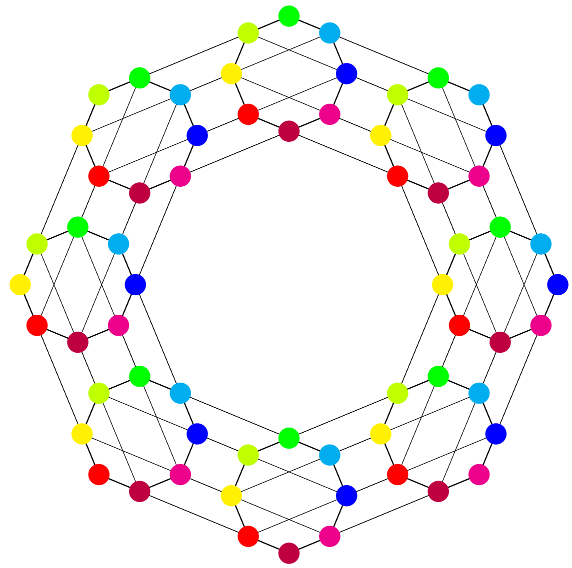
In order to understand why $\gamma_k(K_n \square K_m)$ is important, we have to introduce one fundamental inequality first.

Theorem 2.1. *If G and H are connected graphs with n and m vertices respectively,*

$$\gamma_k(K_n \square K_m) \leq \gamma_k(G \square H).$$

To understand this inequality, we can do some comparison between graphs. For the two graphs below, the one with more edges is $K_8 \square K_8$, and the other one is $C_8 \square C_8$. Fix number k, the smallest k-dominating set of $C_8 \square C_8$ contains either the same number of vertices or more vertices compared to the smallest k-dominating set of $K_8 \square K_8$ since $K_8 \square K_8$ has more adjacency relations between vertices.





And we think inequality probably has some connections with an open conjecture named Vizing's Conjecture.

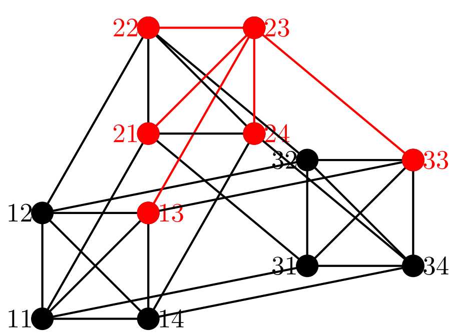
Conjecture 2.1 (Vizing's Conjecture)**.**
For two graphs G and H,

$$\gamma(G)\gamma(H) \leq \gamma(G \square H)$$

The Conjecture is stated by Vadim G. Vizing in 60s. And we already know that $\gamma(K_n \square K_m) \leq \gamma(G \square H)$ when G and H are simple graphs with n and m vertices. Before we try to discover the relation between $\gamma(G \square H)$ and $\gamma(G)\gamma(H)$, the question is how we can analyze k-dominating set on a very complicated Cartesian product of Complete graphs such as $K_8 \square K_8$.

3 Tool

The tool we use is a chessbroad and placing rooks on squares to find k-dominating set. Firtly, we look at one simpler example. The two graphs below are the graph of Cartesian product $K_3 \square K_4$ (On the left) and a 3×4 board (On the right).



11	12	13	14
21	22	23	24
31	32	33	34

For the 3×4 board, each square represents the vertex of $K_3 \square K_4$. We pick vertex 23 specifically. 23 is adjacent to vertex 13 and 33, which are on the one copy of K_3 on the graph on the left and also are on the column where 23 stands on the 3×4 board. At the same time, 23 is adjacent to vertex 21 and 22 and 24, which are on the one copy of K_4 on the graph on the left and also are on the row where 23 occupy on the 3×4 board. It looks like we place one rook on the vertex on the square 23 and all squares on its column and row are adjacent to it. In general, every square follow that pattern and every Cartesian product of Complete graphs, $K_n \square K_m$, can be transferred to a $n \times m$ board. Now we have tools to find 1-dominating set on $K_8 \times K_8$. Let R represent a rook. For $K_8 \square K_8$, we can fill the squares on the first column with rooks and we know that 8 vertices are enough to create a 1-dominating set.

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Based on this property, we can easily visualize the position of every vertex in dominating set and our research is trying to find the smallest number of vertices to k-dominate the Cartesian product of complete graphs. For more information, please wait for our technical report or paper.